

RENORMALONS AND FIXED POINTS*

Georges GRUNBERG

*Centre de Physique Théorique de l'Ecole Polytechnique[†]
91128 Palaiseau Cedex - France*

E-mail: grunberg@orpee.polytechnique.fr

ABSTRACT

The connection between renormalons and power corrections is investigated for the typical infrared renormalon integral assuming the effective coupling constant has an infrared fixed point of an entirely perturbative origin. It is shown that even then the full answer differs from the Borel sum by a power correction. A comparison with the analogue results when the fixed point is generated by the explicit addition of non perturbative power suppressed terms is given.

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It is generally believed that the large order behavior associated to infrared (IR) renormalons, which makes QCD perturbation theory “non Borel summable”, reflects an inconsistency related to the Landau ghost. In this note, I examine how the renormalon problem is resolved when the coupling constant is free of Landau singularity and approaches a non trivial IR fixed point at small momenta. Most of the paper is devoted to the case where the fixed point arises entirely within a perturbative framework, through higher order perturbative corrections. I will show that in this case, IR renormalons are also present, but the exact result differs from the Borel sum by a (complex) power correction, which removes all inconsistencies. At the end, some short comments will be made on the alternative case where the fixed point is generated by the explicit introduction of non perturbative terms.

Let us first review the standard argument^{1,2} for renormalons. Consider the typical IR renormalon integral :

$$R(\alpha) = \int_0^{Q^2} n \frac{dk^2}{k^2} \left(\frac{k^2}{Q^2} \right)^n \alpha_{eff}(k/Q, \alpha) \quad (1)$$

where α is the coupling at scale Q in some arbitrary renormalization scheme, and $\alpha_{eff}(k)$ a renormalization group (RG) invariant effective coupling (I assume $n > 0$, so that the integral in Eq. (1) is IR convergent order by order in perturbation theory). Let us compute the perturbative Borel transform $R(z)$. If $R(\alpha)$ has the formal power series expansion :

$$R_{PT}(\alpha) = \sum_{p=0}^{\infty} r_p \alpha^{p+1},$$

$R(z)$ is defined by :

$$R(z) = \sum_{p=0}^{\infty} \frac{r_p}{p!} z^p \quad (2)$$

The series Eq. (2) are believed to have a finite convergence radius (at the difference of those for $R(\alpha)$), and therefore allow for an all order definition of $R(z)$. One can then *define* an all order “perturbative” resummed $R_{PT}(\alpha)$ by the Borel representation:

$$R_{PT}(\alpha) = \int_0^{\infty} dz \exp\left(-\frac{z}{\alpha}\right) R(z) \quad (3)$$

It is convenient to express $R(z)$ in term of the Borel transform of α_{eff} :

$$\alpha_{eff}(k/Q, \alpha) = \int_0^{\infty} dz \exp\left(-\frac{z}{\alpha}\right) \alpha_{eff}(k/Q, z) \quad (4)$$

Inserting Eq. (4) into Eq. (1), and interchanging the order of integrations, one indeed recovers Eq. (3) with $R(\alpha) = R_{PT}(\alpha)$ and :

$$R(z) = \int_0^{Q^2} n \frac{dk^2}{k^2} \left(\frac{k^2}{Q^2} \right)^n \alpha_{eff}(k/Q, z) \quad (5)$$

IR renormalons arise as an IR divergence ¹ of the integral in Eq. (5), resulting from the IR behavior of $\alpha_{eff}(k/Q, z)$ (which follows solely from the RG invariance of α_{eff}) :

$$\begin{aligned} \alpha_{eff}(k/Q, z) &\simeq \alpha_{eff}(z) \exp \left[-z \left(\beta_0 \ln \left(\frac{k^2}{Q^2} \right) - \frac{\beta_1}{\beta_0} \ln \ln \left(\frac{Q^2}{k^2} \right) \right) \right] \\ k^2 &\ll Q^2 \end{aligned} \quad (6)$$

where β_0 and β_1 are (minus) the one and two loop beta function coefficients. Using Eq. (6) into Eq. (5) one indeed finds for $z \rightarrow z_n n/\beta_0$:

$$R(z) \simeq \alpha_{eff}(z \simeq z_n) \frac{\Gamma(1+\delta)}{n^\delta} \frac{1}{(1 - \frac{z}{z_n})^{1+\delta}} \quad (7)$$

with $\delta = \frac{\beta_1}{\beta_0} z_n$, i.e. $R(z)$ displays a cut singularity ² at the renormalon position $z = z_n$, which generates according to Eq. (3), since $z_n > 0$, an $\mathcal{O}(\exp(-z_n/\alpha))$ imaginary part in $R_{PT}(\alpha)$. In the standard case where α_{eff} has a Landau ghost at some scale $\Lambda^2 < Q^2$, this fact causes no surprise, since the defining integral Eq. (1) itself involves an ambiguous integration over the Landau singularity. But a paradox arises if α_{eff} has an IR fixed point : then $R(\alpha)$ should be perfectly well defined, with no imaginary part according to Eq. (1), whereas Eq. (3), together with Eq. (7), still yield an ambiguous, imaginary amplitude for $R_{PT}(\alpha)$! But Eq. (7) is correct in both cases : as mentionned above, it follows solely from *RG* invariance and the representation Eq. (5) (which is presumably always valid since it is correct order by order in perturbation expansion in z , and the corresponding series are convergent). Thus *IR renormalons are also present in the IR fixed point case*. The only possible conclusion is that Eq. (3) is not in fact a valid representation of Eq. (1), i.e. that $R(\alpha)$ in Eq. (1) does not coincide with $R_{PT}(\alpha)$ in Eq. (3) in the latter case. This is despite the fact I assumed Eq. (4) does instead correctly represents (at least for large enough k) $\alpha_{eff}(k)$ (which is also assumed to have no renormalons), which can thus be determined in principle from “all order” perturbation theory, and qualifies the present framework as being “perturbative” (the alternative possibility that $\alpha_{eff}(k)$ acquires its fixed point through additionnal “higher-twist” type power like corrections not included in the Borel representation will be commented upon below).

Indeed, there is a loophole in the previous derivation that $R(\alpha) = R_{PT}(\alpha)$, which is actually already present when there is a Landau singularity. Assume e.g. α_{eff} is the one loop coupling :

$$\begin{aligned}\alpha_{eff}(k/Q, \alpha) &= \frac{\alpha}{1 + \alpha\beta_0 \ln\left(\frac{k^2}{Q^2}\right)} \\ &\equiv \frac{1}{\beta_0 \ln\left(\frac{k^2}{\Lambda^2}\right)}\end{aligned}$$

Then ¹ : $\alpha_{eff}(k/Q, z) = \exp[-z\beta_0 \ln(k^2/Q^2)]$ and Eq. (5) yields :

$$R(z) = \int_0^{Q^2} n \frac{dk^2}{k^2} \left(\frac{k^2}{Q^2}\right)^n \exp\left[-z \beta_0 \ln\left(\frac{k^2}{Q^2}\right)\right] = \frac{1}{1 - \frac{z}{z_n}}$$

The problem with the previous argument is that Eq. (4) itself is a valid representation of $\alpha_{eff}(k)$ *only if k is not too small*. Indeed in the present example the integral in Eq. (4) converges at $z = \infty$ only if $\frac{1}{\alpha} + \beta_0 \ln\left(\frac{k^2}{Q^2}\right) > 0$, i.e. for $k^2 > \Lambda^2$. For $k^2 < \Lambda^2$, $\alpha_{eff}(k)$ has to be represented instead by a Borel integral over the *negative* z axis :

$$\alpha_{eff}(k/Q, \alpha) = - \int_{-\infty}^0 dz \exp\left(-\frac{z}{\alpha}\right) \alpha_{eff}(k/Q, z) \quad (8)$$

Similar remarks apply in the general case, where Eq. (6) implies, provided $\alpha_{eff}(z)$ decreases no faster than $\exp(-cz)$ at large z , $\alpha_{eff}(k)$ will have a Landau singularity at some scale $k^2 = \Lambda^2$ (see however the fixed point case), below which the Borel representation Eq. (4) will break down, and Eq. (8) should be used instead.

These observations suggest that the correct procedure in the Landau ghost case is to first take $\alpha < 0$ (and slightly complex if $\beta_1 \neq 0$), which implies $Q^2 < \Lambda^2$, so that in the whole integration range in Eq. (1) one has $k^2 < Q^2 < \Lambda^2$, and the representation Eq. (8) is valid *throughout* the range (this means choosing α in the domain of attraction of the *trivial* IR fixed point). Manipulations analogous to those performed above are now justified, and yield :

$$R(\alpha) = - \int_{-\infty}^0 dz \exp\left(-\frac{z}{\alpha}\right) R(z) \quad (9)$$

Finally, $R(\alpha)$ for $\alpha > 0$ (where $Q^2 > \Lambda^2$) is defined as the analytic continuation of Eq. (9), which yields the standard Borel representation Eq. (3) with $R = R_{PT}$.

In the case where $\alpha_{eff}(k)$ has an IR fixed point, a similar problem arises if it happens again that the representation Eq. (4) is not valid below some scale $k = k_{min}$, even if k_{min} does not correspond to a Landau singularity this time. This is possible if, as in the Landau ghost case, $\alpha_{eff}(z)$ does not decrease too fast at large z (the latter assumption seems necessary, because a too fast decrease of $\alpha_{eff}(z)$ (e.g. ¹ if $\alpha_{eff}(z) = \exp(-cz^2)$), although insuring the validity of Eq. (4) for all k 's, usually

yields an $\alpha_{eff}(k)$ which blows up too fast as $k \rightarrow 0$, making the original integral Eq. (1) IR divergent). Furthermore, *and this is the crucial difference with the Landau ghost case*, for $k < k_{min}$, $\alpha_{eff}(k)$, which remains positive, will not admit a Borel representation over the $z < 0$ axis either. (Note that for $k \rightarrow 0$, $\alpha_{eff}(k)$ is always in the strong coupling region, since it approaches a non-trivial fixed point, at the difference of the Landau ghost case, where for $k \rightarrow 0$ one enters an (IR) asymptotically free region below the Landau singularity, since no other fixed point is available). The above derivation that $R = R_{PT}$ can thus not be extended to the IR fixed point case, as expected. Let us now give a general argument that these two functions actually differ by a power correction. Splitting the integration range in Eq. (1) at $k = k_{min}$, one has :

$$\begin{aligned} R(\alpha) &= \int_0^{k_{min}^2} n \frac{dk^2}{k^2} \left(\frac{k^2}{Q^2} \right)^n \alpha_{eff} \left(\frac{k}{Q}, \alpha \right) + \int_{k_{min}^2}^{Q^2} n \frac{dk^2}{k^2} \left(\frac{k^2}{Q^2} \right)^n \alpha_{eff} \left(\frac{k}{Q}, \alpha \right) \\ &\equiv R_- + R_+ \end{aligned}$$

Eq. (4) cannot be used inside the low momentum integral R_- , which can however be parametrized as a power correction since :

$$\begin{aligned} R_- &= \left(\frac{k_{min}^2}{Q^2} \right)^n \int_0^{k_{min}^2} n \frac{dk^2}{k^2} \left(\frac{k^2}{k_{min}^2} \right)^n \alpha_{eff} \left(\frac{k}{k_{min}}, \alpha_{min} \right) \\ &\equiv \left(\frac{k_{min}^2}{Q^2} \right)^n R(\alpha_{min}) \end{aligned}$$

where $\alpha_{min} = \alpha(Q = k_{min})$, and RG invariance has been used in the first step. On the other hand, using Eq. (4) in R_+ , one gets :

$$R_+ = \int_0^\infty dz \exp\left(-\frac{z}{\alpha}\right) \int_{k_{min}^2}^{Q^2} n \frac{dk^2}{k^2} \left(\frac{k^2}{Q^2} \right)^n \alpha_{eff} \left(\frac{k}{Q}, z \right)$$

To go further, it is necessary to know the k -dependence of $\alpha_{eff}(k/Q, z)$. This can be done easily in the special case where $\beta_1 = 0$, if one chooses $\alpha(Q)$ to be the one loop coupling. Then one can show¹ that : $\alpha_{eff}(k/Q, z) = \alpha_{eff}(z) \exp[-z\beta_0 \ln(k^2/Q^2)]$, and one gets :

$$\int_{k_{min}^2}^{Q^2} n \frac{dk^2}{k^2} \left(\frac{k^2}{Q^2} \right)^n \alpha_{eff} \left(\frac{k}{Q}, z \right) = \alpha_{eff}(z) \frac{1}{1 - \frac{z}{z_n}} \left[1 - \left(\frac{k_{min}^2}{Q^2} \right)^{z_n \beta_0 (1 - z/z_n)} \right]$$

Hence :

$$\begin{aligned}
R_+ &= \int_0^\infty dz \exp\left(-\frac{z}{\alpha}\right) \alpha_{eff}(z) \frac{1}{1 - \frac{z}{z_n}} - \left(\frac{k_{min}^2}{Q^2}\right)^n \int_0^\infty dz \exp\left(-\frac{z}{\alpha_{min}}\right) \alpha_{eff}(z) \frac{1}{1 - \frac{z}{z_n}} \\
&\equiv R_{PT}(\alpha) - \left(\frac{k_{min}^2}{Q^2}\right)^n R_{PT}(\alpha_{min})
\end{aligned}$$

One therefore ends up with the result :

$$R(\alpha) = R_{PT}(\alpha) + \left(\frac{k_{min}^2}{Q^2}\right)^n [R(\alpha_{min}) - R_{PT}(\alpha_{min})] \quad (10)$$

where the coefficient of the power correction is given by the discrepancy between the exact amplitude and its Borel representation. Eq. (10) is equivalent to the statement that $R(\alpha) = R_{PT}(\alpha) + const/(Q^2)^n$, and is also correct when $\beta_1 \neq 0$ (see below). Note it is not possible the power correction vanishes (as it does in the Landau ghost case) since $R(\alpha_{min})$ is real, whereas $R_{PT}(\alpha_{min})$ is complex due to the effect of the renormalon (the power correction must be complex to cancel the imaginary part of $R_{PT}(\alpha)$).

I now illustrate the previous discussion with the example of the 2 loop coupling :

$$\frac{d\alpha_{eff}}{d\ln k^2} = -\beta_0(\alpha_{eff})^2 - \beta_1(\alpha_{eff})^3 \quad (11)$$

I shall consider both the standard case $\beta_1/\beta_0 > 0$ where there is a Landau singularity, and the case $\beta_1/\beta_0 < 0$ where $\alpha_{eff}(k)$ has an IR fixed point at $\alpha_{IR} = \beta_0/\beta_1$ (which actually occurs in QCD for a large enough number of flavors). Remarkably, $R(z)$ can be computed *exactly* with a straightforward change of variable, adapted from a similar one suggested in ref.3. Defining the Borel variable by :

$$\frac{z}{z_n} = \frac{1 - \frac{\alpha}{\alpha_{eff}(k)}}{1 + \frac{\beta_1}{\beta_0}\alpha} \quad (12)$$

with $\alpha \equiv \alpha_{eff}(k = Q)$, and using the solution of Eq. (11) :

$$\ln\left(\frac{k^2}{\Lambda^2}\right) = \frac{1}{\beta_0\alpha_{eff}} - \frac{\beta_1}{\beta_0^2} \ln\left(\frac{1}{\alpha_{eff}} + \frac{\beta_1}{\beta_0}\right) + \frac{\beta_1}{\beta_0^2} \quad (13)$$

(together with the similar relation at $k = Q$) one obtains :

$$n \frac{dk^2}{k^2} \left(\frac{k^2}{Q^2}\right)^n \alpha_{eff}(k) = -dz \frac{\exp(-\frac{z}{\alpha} - \frac{\beta_1}{\beta_0}z)}{\left(1 - \frac{z}{z_n}\right)^{1+\delta}}$$

which suggests the looked for Borel transform is :

$$R(z) = \frac{\exp\left(-\frac{\beta_1}{\beta_0}z\right)}{\left(1 - \frac{z}{z_n}\right)^{1+\delta}}. \quad (14)$$

To complete the proof, it remains to determine the new integration bounds.

1) Assume first $\beta_1/\beta_0 > 0$: then α_{eff} has a Landau singularity at the scale $\bar{\Lambda}^2 = \Lambda^2 \exp\left(-\frac{\beta_1}{\beta_0^2} \ln\left(\frac{\beta_1}{\beta_0}\right) + \frac{\beta_1}{\beta_0^2}\right)$. I assume $Q^2 > \bar{\Lambda}^2$, so that $\alpha > 0$. For $k^2 = Q^2$, $\alpha_{eff} = \alpha$, and $z = 0$; at $k^2 = \bar{\Lambda}^2$, $\alpha_{eff} = \infty$, and $z = z_L \equiv \frac{z_n}{1 + \frac{\beta_1}{\beta_0}\alpha} < z_n$; decreasing k^2 below $\bar{\Lambda}^2$, z becomes complex ; finally, for $k^2 \rightarrow 0$, $\alpha_{eff} \rightarrow 0^-$, and $z \rightarrow +\infty$. $R(\alpha)$ in Eq. (1) then takes the form of a Borel integral along a path in the complex z -plane, which divides into two complex conjuguate branches at $z = z_L$, and can be deformed to above or below the positive real axis to yield the standard Borel representation Eq. (3) with $R(z)$ as in Eq. (14).

2) On the other hand, if $\beta_1/\beta_0 < 0$, the situation is actually simpler : as k^2 decreases from Q^2 to 0, α_{eff} increases from α to the IR fixed point α_{IR} , and z increases from 0 to z_n through real values so that Eq. (1) becomes :

$$R(\alpha) = \int_0^{z_n} dz \exp\left(-\frac{z}{\alpha}\right) \frac{\exp\left(-\frac{\beta_1}{\beta_0}z\right)}{\left(1 - \frac{z}{z_n}\right)^{1+\delta}} \quad (15)$$

(the renormalon singularity is integrable at $z = z_n$, since $\delta < 0$ now). We therefore check that $R(\alpha)$ is *not* given by the Borel sum $R_{PT}(\alpha)$ in the IR fixed point case. Rather, it is given by $R_{PT}(\alpha)$ minus an *exponentially small* $\mathcal{O}(\exp(-z_n/\alpha))$ correction :

$$\begin{aligned} R(\alpha) &= \int_0^\infty dz \exp\left(-\frac{z}{\alpha}\right) \frac{\exp\left(-\frac{\beta_1}{\beta_0}z\right)}{\left(1 - \frac{z}{z_n}\right)^{1+\delta}} - \int_{z_n}^\infty dz \exp\left(-\frac{z}{\alpha}\right) \frac{\exp\left(-\frac{\beta_1}{\beta_0}z\right)}{\left(1 - \frac{z}{z_n}\right)^{1+\delta}} \\ &\equiv R_{PT} + R_{NP} \end{aligned} \quad (16)$$

The latter is just a (complex) power correction, in accordance with the general expectation. Indeed, Eq. (15)-(16) simplify by performing the change of coupling : $1/a = 1/\alpha + \beta_1/\beta_0$, and one gets in particular :

$$\begin{aligned}
R_{NP} &= - \int_{z_n}^{\infty} dz \exp \left(-\frac{z}{a} \right) \frac{1}{\left(1 - \frac{z}{z_n}\right)^{1+\delta}} = -\tilde{C} \exp(-z_n/a)(-1/a)^\delta \\
&= -\tilde{C}(-1)^\delta (\Lambda^2/Q^2)^n
\end{aligned} \tag{17}$$

with $\tilde{C} = \frac{\beta_0}{\beta_1} \Gamma(1-\delta)(z_n)^\delta$, and Eq. (13) at $k = Q$ was used in the last step (in this example, the subtracted term $-R_{NP}$ corresponds exactly to the “minimal” prescription of ref.4 to “regularize” IR renormalons). The same method can deal with the case $\alpha < 0$, where one is in the domain of attraction of the *trivial* IR fixed point (provided the condition $1 + \frac{\beta_1}{\beta_0} \alpha > 0$ is also satisfied if $\beta_1/\beta_0 < 0$). Then $\alpha_{eff}(k)$ monotonously increases from α to 0^- as k^2 decreases from Q^2 to 0, hence $z < 0$ and decreases from 0 to $-\infty$. Thus :

$$R(\alpha) = - \int_{-\infty}^0 dz \exp \left(-\frac{z}{\alpha} \right) \frac{\exp \left(-\frac{\beta_1}{\beta_0} z \right)}{\left(1 - \frac{z}{z_n}\right)^{1+\delta}} \tag{18}$$

i.e. a Borel integral over the *negative* z axis, in agreement with Eq. (9) (note that Eq. (15) is *not* the analytic continuation of Eq. (18)).

For completeness I give also the result if $\alpha_{eff}(k)$ satisfies the RG equation :

$$\frac{d\alpha_{eff}}{d \ln k^2} = \frac{-\beta_0 \alpha_{eff}^2}{1 - \frac{\beta_1}{\beta_0} \alpha_{eff}}$$

where the *inverse* beta function has only two terms. The appropriate change of variable in this case turns out to be precisely the one suggested in ref.3 :

$$\frac{z}{z_n} = 1 - \frac{\alpha}{\alpha_{eff}(k)}$$

(with $\alpha \equiv \alpha_{eff}(k = Q)$), which yields for $\beta_1/\beta_0 > 0$ (using also the convolution theorem) :

$$R(\alpha) = \frac{\delta}{1+\delta} \alpha + \frac{1}{1+\delta} \int_0^\infty dz \exp \left(-\frac{z}{\alpha} \right) \frac{1}{\left(1 - \frac{z}{z_n}\right)^{1+\delta}}$$

whereas for $\beta_1/\beta_0 < 0$, where α_{eff} has an *infinite* IR fixed point, one gets :

$$R(\alpha) = \frac{\delta}{1+\delta} \alpha + \frac{1}{1+\delta} \int_0^{z_n} dz \exp \left(-\frac{z}{\alpha} \right) \frac{1}{\left(1 - \frac{z}{z_n}\right)^{1+\delta}}$$

and shows that in this case too (with a different \tilde{C}) : $R(\alpha) = R_{PT}(\alpha) - \tilde{C}(-1)^\delta (\Lambda^2/Q^2)^n$.

For an arbitrary β_{eff} function with an IR fixed point one expects however the answer to be of the more general form :

$$R(\alpha) = R_{PT}(\alpha) + \left(\frac{\Lambda^2}{Q^2}\right)^n (-\tilde{C}(-1)^\delta + C) \quad (19)$$

where C and \tilde{C} are real, and independent of Q , and \tilde{C} is proportionnal to the renormalon residue. It may be worth mentionning that taking the Q^2 derivative of both sides of Eq. (19), and going to Borel space, one easily derives a relation between $R(z)$ and the Borel transform of $\beta_{eff}(\alpha) = d\alpha/d\ln Q^2$ (with $\alpha \equiv \alpha_{eff}(k = Q)$) :

$$R(z) - \frac{\beta_0}{n} z R(z) - \frac{1}{n} \int_0^z dy b(z-y)y R(y) = 1 \quad (20)$$

where $b(z)$ is the Borel transform of $b(\alpha) \equiv -\frac{\beta_{eff}(\alpha)}{\alpha^2} - \beta_0$. Eq. (20) can be used to rederive the previous two results, and allows to deal with more complicated examples as well. Note also that C and \tilde{C} , being independent of Q , have dropped from Eq. (20), and in fact any Q -dependence in these coefficients would be inconsistent with the assumption of the perturbative nature of the coupling (equivalent here to assuming that $\beta_{eff}(\alpha)$ coincides with its Borel sum).

Non perturbative IR fixed point: in the Landau ghost case, one has to add explicitly power suppressed non perturbative terms α_{NP} to remove the Landau ghost and generate an IR fixed point. Putting $\alpha_{eff} = \alpha_{PT} + \alpha_{NP}$, where α_{PT} is assumed to have a Landau singularity and to satisfy the Borel representation Eq. (4) we thus have: $R(Q)R_{PT}(Q) + R_{NP}(Q)$ where $R_{PT}(Q)$ is given by Eq. (3), and:

$$\begin{aligned} R_{NP}(Q) &= \int_0^{Q^2} n \frac{dk^2}{k^2} \left(\frac{k^2}{Q^2}\right)^n \alpha_{NP}(k) \\ &= \left(\frac{\Lambda^2}{Q^2}\right)^n [-\tilde{C}(-1)^\delta + C(Q)] \end{aligned} \quad (21)$$

where $C(Q)$ is real and represents the part of the power corrections unrelated to renormalons. To make contact with ref.5, I now assume the integral in Eq. (21) converges at infinity, in which case $C(Q) \rightarrow C$ at large Q , with :

$$-\tilde{C}(-1)^\delta + C = \int_0^\infty n \frac{dk^2}{k^2} \left(\frac{k^2}{\Lambda^2}\right)^n \alpha_{NP}(k) \quad (22)$$

and, neglecting higher order power corrections :

$$R(Q) \simeq R_{PT}(Q) + \left(\frac{\Lambda^2}{Q^2}\right)^n (-\tilde{C}(-1)^\delta + C) \quad (23)$$

Assume now the high energy approximation Eq. (23) is valid for $Q \gtrsim \mu_I$. This is equivalent to assume that the high energy tail of the integral in Eq. (22) can be

neglected for $k \gtrsim \mu_I$. Then from Eq. (23) and its counterpart at $Q = \mu_I$ one gets immediately:

$$R(Q) \simeq R_{PT}(Q) + \left(\frac{\mu_I^2}{Q^2}\right)^n [R(\mu_I) - R_{PT}(\mu_I)] \quad (24)$$

which is the parametrisation of the power corrections in term of low energy integrals over the full (IR regular) and the perturbative effective couplings suggested in ref.5. In this approach, the renormalon problem is completely bypassed, since Eq. (24) can be equivalently written as:

$$R(Q) \simeq \int_{\mu_I^2}^{Q^2} n \frac{dk^2}{k^2} \left(\frac{k^2}{Q^2}\right)^n \alpha_{PT}\left(\frac{k}{Q}, \alpha\right) + \left(\frac{\mu_I^2}{Q^2}\right)^n R(\mu_I) \quad (25)$$

and the integral over α_{PT} , being cutoff in the infrared, has no renormalons. Perturbation theory, if used to estimate the integral in Eq. (25), is then expected to be convergent at arbitrary high order. Eq. (25) amounts to neglect $\alpha_{NP}(k)$ for $k \gtrsim \mu_I$, which is the assumption of ref.5. We have seen in the example of the integral of Eq. (1) this assumption can be justified if $\alpha_{NP}(k)$ decreases sufficiently fast at large k . Furthermore we have shown that Eq. (23) and (24) are actually *exact* (Eq. (10) and (19)) in the case where $\alpha_{NP} = 0$ and the fixed point is of perturbative origin (then Eq. (25) is trivial), provided one interprets $R_{PT}(Q)$ as the Borel sum of perturbation theory.

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